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FUNDAMENTAL PRINCIPLES IN THE BUCKLING OF STRUCTURES UNDER COMBINED LOADING

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Abstract—The basic concepts of the general theory of elastic stability are developed with reference to combined loading, only discrete conservative structural systems being considered.

It is observed that the two well known critical points, the limit and bifurcation points, are not quite adequate to describe the buckling behaviour of structures under combined loading. A more appropriate classification which describes the nature of buckling more aptly is given. Thus, in the elastic stability of structures under combined loading, mainly two types of critical point are involved, the "general" and "special" critical points. The conditions giving rise to these two distinct phenomena and their relation to the limit and bifurcation points are examined in detail. The connexion between the shape of the equilibrium surface and the nature of buckling is demonstrated analytically.

1. INTRODUCTION

THE foundations of the general theory of elastic stability were laid by Poincaré [1] in his classical paper on the stability of rotating liquid masses. Using the concept of generalized coordinates, he showed that a loss of stability is normally associated with either a limit point at which an initially stable equilibrium path reaches a local extremum or with a point of bifurcation at which the path intersects a second and distinct equilibrium path. The significance of these two critical conditions in the subject of elastic stability is clear, and both conditions have later been examined by a few authors in great detail.

Koiter [2, 3], working in the context of continuum mechanics focused his attention on the branching configurations, while Lyttleton [4] and Thompson [5] discussed more general critical configurations associated with a limit point as well as a bifurcation point in terms of generalized coordinates. The latter author demonstrated the essential inter-relationship between the two distinct phenomena analytically. His results show that limit points are—mathematically—more general and arise when a single stability coefficient vanishes, while the bifurcation points arise when a stability coefficient and a second energy coefficient vanish simultaneously.

The structural systems considered by these authors involve a single variable loading parameter which is assumed to describe the external loading of the structure entirely. In the presence of more than one independent loading parameter, however, the situation is not quite the same due to the involvement of an equilibrium surface which is defined [6, 7] in the load–deflection space as the entirety of the equilibrium points. It is observed [6] that, depending on the shape of the equilibrium surface, a limit point can, under some circumstances, also be regarded as a point of bifurcation. This is an interesting result which might lead to confusion as to the type of the critical point and the nature of buckling. In addition, the basic properties of the stability boundary, which comes into picture in

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connection with the equilibrium surface (which is defined [6] in the loading-space as the entirety of the critical points associated with an initial loss of stability), depend on the type of the critical point. It is, therefore, felt that a careful study of the fundamental aspects of the buckling behaviour of structures under combined loading is essential. In the present paper this is done in terms of generalized coordinates, and a reclassification of the critical points together with the conditions giving rise to these points is presented.

Only conservative elastic systems are considered.

2. THE STRUCTURAL SYSTEM

We consider a conservative elastic structural system characterized by a total potential energy function

$$V = V(Q_i, \Lambda^j) \tag{1}$$

which is assumed to be single-valued and well-behaved at least in the region of interest. The $Q_i (i = 1, 2, \ldots, N)$ are the generalized coordinates, and we assume that a given set of the Q_i defines completely the configuration of the system. The independent loading parameters $\Lambda^j (j = 1, 2, \ldots, M)$ might represent generalized forces, the magnitudes of the external loads and even imperfections.

The necessary and sufficient condition of equilibrium, that the first variation of the potential energy with respect to the Q_i should vanish, yields a set of N equilibrium equations

$$\frac{\partial V(Q_i, \Lambda^j)}{\partial Q_k} = 0 \tag{2}$$

which define an M-dimensional equilibrium surface in the $M+N$ dimensional load-deflection space.

To analyse an element of such a surface, we can expand the total potential energy function (1) as a Taylor series about an equilibrium state of interest. Suppose the equations (2) are solved simultaneously to yield solutions in the form

$$Q_i = Q_i(\Lambda^j), \qquad i = 1, \ldots, N; \qquad j = 1, \ldots, M. \tag{3}$$

Consider an arbitrarily chosen point F on the surface representing a state of equilibrium $Q_i^F(\Lambda_F^j)$ which will be called fundamental. Using q_i and λ^j to denote increments in the variables Q_i and Λ^j respectively, we shall refer the potential energy of the system to the fundamental state by writing it in the form

$$V = V[Q_i^F + q_i, \Lambda_F^j + \lambda^j]. \tag{4}$$

We further introduce a linear, non-singular and orthogonal transformation

$$q_i = \sum_{j=1}^{N} \alpha_{ij} u_j, \qquad |\alpha_{ij}| \neq 0, \tag{5}$$

to diagonolize the quadratic form (in q_i) of the energy expansion around the fundamental state. By insisting that the transformation matrix $[\alpha_{ij}]$ should be orthogonal we, in general, ensure a unique point transformation which changes the coordinate axes from the q_i to the u_i, the latter being orthogonal through the origin of the N-dimensional q_i coordinate system [i.e. through the point F (see Fig. 1)].

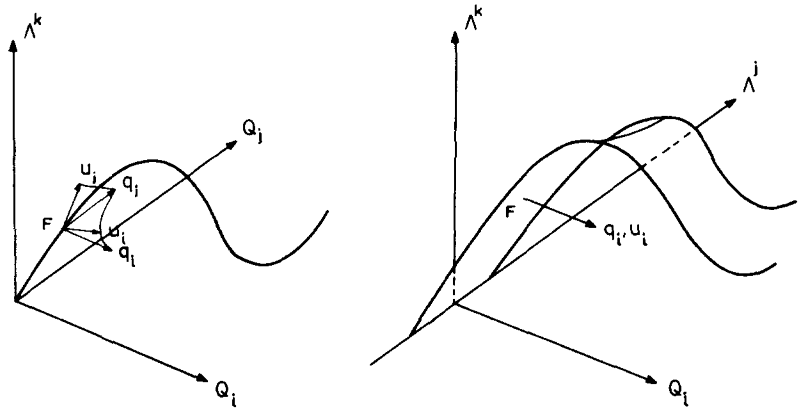

FIG. 1.

Introducing the transformation (5) into the equation (4) we get a new energy function

$$H(u_i, \lambda^j) \equiv V[Q_i^F + \alpha_{ij}u_j, \Lambda_F^k + \lambda^k]. \tag{6}$$

Using the Taylor's expansion and remembering that the fundamental state is one of equilibrium we get

$$\begin{aligned} H = {} & H_0 + H^i\lambda^i + \frac{1}{2!}(H_{ii}u_i^2 + 2H_i^ju_i\lambda^j + H^{ij}\lambda^i\lambda^j) \\ & + \frac{1}{3!}(H_{ijk}u_iu_ju_k + 3H_{ij}^ku_iu_j\lambda^k + 3H_i^{jk}u_i\lambda^j\lambda^k + H^{ijk}\lambda^i\lambda^j\lambda^k) \\ & + \frac{1}{4!}(\cdots\cdots) + \frac{1}{5!}(\cdots\cdots) + \cdots \end{aligned} \tag{7}$$

Here and in the remainder of this paper the summation convention is adopted both for the subscripts and superscripts separately so that summations are taken over repeated superscripts from 1 to M and over repeated subscripts from 1 to N. It is also understood that the upper and lower indices on the H coefficients denote partial differentiation with respect to the corresponding loading parameter and generalized coordinate respectively, all derivatives being evaluated at the fundamental state of equilibrium F.

3. EQUILIBRIUM SURFACE

We are now in a position to investigate the shape of the equilibrium surface in the vicinity of the fundamental state F. In general, all the coefficients are finite and non-zero in which case the N equilibrium equations $\partial H/\partial u_i$ yield to a first approximation

$$H_{ii}u_i + \mathrm{H}_i^j\lambda^j = 0. \tag{8}$$

These equations determine an M-dimensional plane in the $M+N$ dimensional load–deflection space and indicate a one-to-one correspondence between a set of loading parameters and a set of generalized coordinates. This is the most general behaviour associated with the equilibrium of the system.

For the stability of an equilibrium state a sufficient condition is that the potential energy has a complete relative minimum at that point. It immediately follows that the positive definiteness of the quadratic form of the energy is a sufficient condition of stability. If the quadratic form admits negative values the associated equilibrium state is unstable. If the quadratic form admits zero values, then, the system is said to be in a critical (or neutral) state of equilibrium. Since the quadratic form in q_i has already been diagonolized by the linear transformation (5), at least one of the Poincaré's stability coefficients $H_{ii}(i = 1, \ldots, N)$ should vanish if the fundamental state is critical. If we suppose that the stability coefficients are arranged in the descending order

$$H_{11} \leq H_{22} \leq \ldots \leq H_{NN}:$$

we can, then, discuss the stability of the equilibrium state simply by referring to H_{11} only. Thus, for $H_{11} > 0$ the state is stable while for $H_{11} < 0$ the state is unstable. $H_{11} = 0$ is the only critical case in which the stability of the state cannot be determined and higher order variations of energy are required.

After this brief general discussion of stability of a certain equilibrium state we can now return to our analysis and suppose that the fundamental state F is moved on the equilibrium surface until it coincides with a discrete critical point where, say $H_{11} = 0$ and $H_{ss} \neq 0$ $(s \neq 1)$. It is understood that the above arrangement of H_{ii} in the descending order was introduced merely to facilitate the discussion of stability of a certain equilibrium state, and is not meant to be valid here and in the following analyses. Now, the most important factor which appears to determine the shape of the equilibrium surface and the type of buckling is $\overrightarrow{\text{grad}}_\lambda\, H_1(0, 0)$. Two interesting cases arise and we shall study them separately:

(a) *The case in which* $\text{grad}_\lambda\, H_1(0, 0) \neq 0$.

This implies that at least one of the coefficients $H_1^i(0, 0)$ in the energy expansion (7) does not vanish. We shall call the critical point "general" if this condition is satisfied, and if, in addition, the non-vanishing stability coefficients (H_{ss}) are all positive it will be called "general primary".

Supposing, for now, that all the coefficients are non-zero, the equilibrium equations can be written down in the form

$$\frac{\partial H}{\partial u_1} = H_1^i \lambda^i + \tfrac{1}{2} H_{1ij} u_i u_j + \ldots = 0 \tag{9}$$

and

$$\frac{\partial H}{\partial u_s} = H_{ss} u_s + H_s^i \lambda^i + \tfrac{1}{2} H_{s11} u_1^2 + \ldots = 0 \tag{10}$$

which define an M-dimensional equilibrium surface in the $M+N$ dimensional load-deflection space. Substituting for u_s in the equation (9) and keeping to a first approximation we get

$$H_1^i \lambda^i + \tfrac{1}{2} H_{111} u_1^2 = 0, \tag{11}$$

and substituting for u_1 in the equation (10) we have

$$H_{ss} u_s + \left(H_s^i - \frac{H_1^i H_{s11}}{H_{111}} \right) \lambda^i = 0. \tag{12}$$

Equations (11) and (12) can be regarded as the projections of the equilibrium surface into the $u_1 - \lambda^i$ and $u_s - \lambda^j$ subspaces respectively, the former representing a curved surface and the latter a plane. In other words the critical coordinate u_1 ceases to be single-valued while the non-critical coordinates, similar to the case of a non-critical equilibrium point, remain as linear functions of the loading parameters. These projections are shown schematically in Figs. 2 and 3.

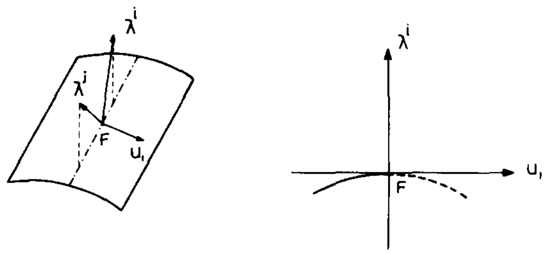

FIG. 2.

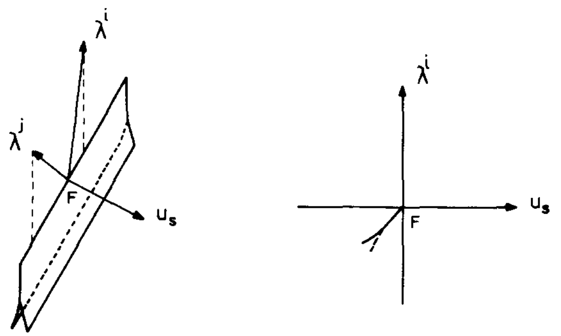

FIG. 3.

It is interesting to note that the equations (11) and (12) are similar to those obtained by Thompson [5] for a limit point who used a single loading parameter, the only difference being the summation on the loading parameters. In fact one can readily derive Thompson's results on the basis of these equations by assuming that the $\lambda^i(i = 1, \cdots M)$ are functions of a single variable parameter ξ. Expanding the functions $\lambda^i(\xi)$ around the point F where $\lambda^i = \xi = 0$ we get

$$\lambda^i = l^i\xi + k^i\xi^2 + \ldots \tag{13}$$

in which $l^i, k^i, \ldots$ are constants.

Substituting for the λ^i in the equation (11), the first order result can be written as

$$H_1^i l^i \xi + \tfrac{1}{2} H_{111} u_1^2 = 0 \tag{14}$$

in which the summation $H_1^i l^i$ is immediately recognized as $H_1^\xi (\equiv \partial H_1/\partial \xi)$ evaluated at $u_i = \lambda^i = \xi = 0$ giving finally

$$H_1^\xi \xi + \tfrac{1}{2} H_{111} u_1^2 = 0 \tag{15}$$

which is, indeed, the relationship obtained by Thompson for a limit point. Similarly the equation (12) can also be converted to Thompson's corresponding expression.

An important distinction between the systems with a single loading parameter and those with several parameters can now be drawn. Thus, in contrast to the situation considered by Thompson, the non-vanishing energy coefficients (H^i_1) do not here ensure a limit point since the summation $H^i_1 l^i$ might vanish depending on the l^i (i.e. on the loading) in which case the equilibrium equations must be reconsidered and additional terms retained. If this is done we obtain to a first approximation

$$\tfrac{1}{2}H_{111}u_1^2 + au_1\xi + \tfrac{1}{2}b\xi^2 = 0 \tag{16}$$

and

$$H_{ss}u_s + H^i_s l^i \xi = 0 \tag{17}$$

where a and b are constants and given by

$$\left.\begin{aligned} a &= \left(H^i_{11} - H_{11s}\frac{H^i_s}{H_{ss}}\right) l^i \equiv c^i l^i \\ b &= \left(H^{ij}_1 + H_{1sr}\frac{H^i_s H^j_r}{H_{ss}H_{rr}} - 2H^i_{1s}\frac{H^j_s}{H_{ss}}\right) l^i l^j + 2H^i_1 k^i \\ &\equiv d^{ij} l^i l^j + 2H^i_1 k^i. \end{aligned}\right\} \tag{18}$$

The equation (16) can be solved for u_1 to yield

$$u_1 = \frac{1}{H_{111}}[-a \pm (a^2 - H_{111}b)^{\frac{1}{2}}]\xi \tag{19}$$

which indicates bifurcation on a plot of u_1 against ξ (Fig. 4) provided

$$a^2 - H_{111}b > 0. \tag{20}$$

This phenomenon can be seen in another way. Suppose some of the coefficients H^i_1 vanish (say $H^t_1 = 0$ where t takes values from 1 to M_1, $M_1 < M$), then, the equilibrium equations can be written as

$$H^x_1\lambda^x + \tfrac{1}{2}H_{1ij}u_i u_j + H^t_{1j}u_j\lambda^t + \tfrac{1}{2}H^{pt}_1\lambda^p\lambda^t + \ldots = 0 \tag{21}$$

and

$$H_{ss}u_s + H^i_s\lambda^i + \tfrac{1}{2}H_{s11}u_1^2 + \ldots = 0 \tag{22}$$

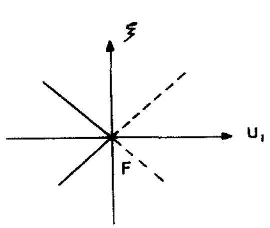

FIG. 4.

where $s \neq 1$, t and p range from 1 to M_1, x ranges from M_1+1 to M (i.e. $x \neq t$, $x \neq p$), and summation is carried out on repeated subscripts and superscripts over admissible ranges.

Substituting for u_s in the equation (21) and keeping to a first approximation we get

$$H_1^x\lambda^x + \tfrac{1}{2}H_{111}u_1^2 + c^t u_1\lambda^t + d^{tp}\lambda^t\lambda^p = 0, \tag{23}$$

and substituting for u_1 in the equation (22) we have

$$H_{ss}u_s + \left(H_s^x - \frac{H_1^x H_{s11}}{H_{111}}\right)\lambda^x + H_s^t\lambda^t = 0 \tag{24}$$

where $s \neq 1$, $x \neq t$, $x \neq p$ and the coefficients c^t and d^{tp} are given by the equations (18) (Note that the summation range is now from 1 to $M_1, M_1 < M$).

We now see clearly that bifurcation is not ruled out. In fact, if we take a ray determined by

$$\left.\begin{aligned} \lambda^x &= 0 \\ \lambda^t &= l^t\xi, \qquad t \neq x \end{aligned}\right\} \tag{25}$$

where l^t are the direction cosines and ξ the radius vector, the equilibrium equation (23) yields

$$\tfrac{1}{2}H_{111}u_1^2 + Cu_1\xi + \tfrac{1}{2}D\xi^2 = 0 \tag{26}$$

in which $C = c^t l^t$ and $D = d^{tp}l^t l^p$. Solution of (26) for u_1 indicates bifurcation as illustrated in Fig. 4 provided $C^2 - H_{111}D > 0$.

On the other hand, suppose we take a general ray in the sense that all the $l^i \neq 0$ (or at least some of $l^x \neq 0$); the equation (23) will then yield

$$H_1^x l^x\xi + \tfrac{1}{2}H_{111}u_1^2 = 0 \tag{27}$$

provided $H_1^x l^x \neq 0$, indicating a limit point (Fig. 5). In the event that $H_1^x l^x = 0$ we return to the case considered before and find that bifurcation instead of a limit point is involved.

It is thus demonstrated analytically that a "general" critical point can be considered both as a limit and bifurcation point depending on the mode of loading.

We shall now examine the second case,

(b) *The case in which* $\overrightarrow{\mathrm{grad}}_\lambda\, H_1(0,0) = 0$

This condition implies that all the coefficients $H_1^i (i = 1, \ldots, M)$ vanish at the fundamental state which was chosen as a discrete critical point where $H_{11} = 0$ and all $H_{ss} \neq 0$ for $s \neq 1$. This is of course a special critical state compared to the more general case (a),

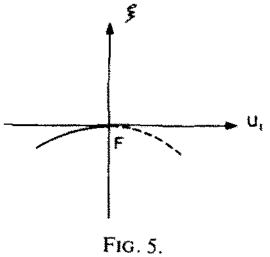

FIG. 5.

and it will be called "special". Assuming further that all the other energy coefficients are non-zero, the first order solution of the equilibrium equations yield

$$\tfrac{1}{2}H_{111}u_1^2+c^iu_1\lambda^i+\tfrac{1}{2}d^{ij}\lambda^i\lambda^j = 0 \tag{28}$$

and

$$H_{ss}u_s+H_s^i\lambda^i = 0 \tag{29}$$

where c^i and d^{ij} are defined by the equations (18)..Let us take an arbitrary ray determined by $\lambda^i = l^i\xi$ where some $l^i \neq 0$. Substituting for the λ^i in the equation (28) and solving for u_1 we get

$$u_1 = \frac{1}{H_{111}}[-C\pm(C^2-H_{111}D)^{\frac{1}{2}}]\xi \tag{30}$$

where $C = c^il^i$ and $D = d^{ij}l^il^j$. We see that the solution involves a point of bifurcation provided $C^2-H_{111}D > 0$. We also note that limit points are now definitely ruled out. i.e. There exist no rays with respect to which the critical point F can be regarded as a limit point. Thus, the conditions which ensure bifurcation buckling and exclude the possibility of snap-buckling are supplied by the vanishing of $\overrightarrow{\text{grad}}_\lambda\, H_1(0,0)$ together with $C^2-H_{111}D > 0$.

It can readily be shown that the particular case in which all the coefficients $H_i^j(i = 1,\ldots,N;j = 1,\ldots,M)$ vanish is also associated with a "special" critical point at which snap-buckling is excluded.

4. CONCLUSIONS

It is shown that the loss of stability of a structural system with independent loading parameters can be associated with either a "general" or a "special" critical point. The former arises when the gradient of $\partial H/\partial u_1$ with respect to the loading parameters is not zero and the projection of the equilibrium surface in the u_1–λ^i subspace is, then, continuous. The latter arises when this gradient vanishes in which case the intersection of two surfaces is involved.

Although a "general" critical point is normally associated with a limit point, it is demonstrated analytically that, under some conditions, bifurcation of solution can also occur at the same point. A "special" critical point, on the other hand, is a genuine bifurcation point at which a simple extremum is definitely ruled out.

In demonstrating the connexion between the shape of the equilibrium surface and the nature of buckling, the analysis is facilitated by the use of certain rays in the form $\lambda^i = l^i\xi$.

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Абстракт—Даются основные концепции общей теории упругой устойчивости по отношению к комбинированной нагрузке, причем рассматриваются только дискретные консервативные системы конструкций.

Наблюдается, что хорошо известные критические точки, т.е. точка предела и бифуркации, не пригодны для описания поведения потери устойчивости конструкций под влиянием комбинированной нагрузки. Приводится более подходящая классификация, более полно описывающая природу выпучивания. Рассматриваются, в области устойчивости упругих систем под влиянием комбинированной нагрузкм, главным образом два типа критических точек—"общая" и "специальная" точки. Исследуется, детально, условия, которые приводят к этим двум отдельным явлениям и их зависимость от точек предела и бифуркации. Представляется, аналитически, зависимость между формой поверхности равновесия и природой потери устойчивости.